Lecture 12

Lec 12

Implicit Differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another example - for example :

$y = \sqrt{x^2+1}$, $y = \sin x$ or in general , $y = f(x)$

So functions , however , are defined implicitly by a relation betn x & y such as

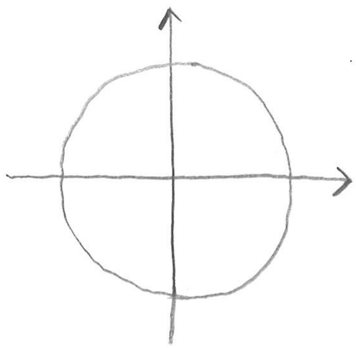
- $x^2 + y^2 = 25 \longrightarrow 1$
- $x^3 + y^3 = 6xy$

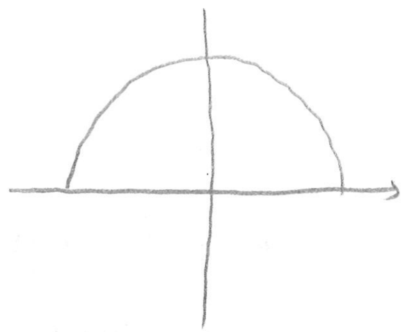
In some cases it is possible to solve such an equation for y as an explicit or several functions of x .

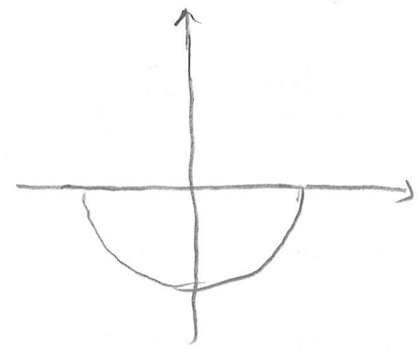
For instance , $x^2 + y^2 = 25 \Rightarrow y^2 = 25 - x^2 \Rightarrow y = \pm\sqrt{25-x^2}$

So two of the functions determined the implicit equation 1 are

$f(x) = \sqrt{25-x^2}$, $g(x) = -\sqrt{25-x^2}$







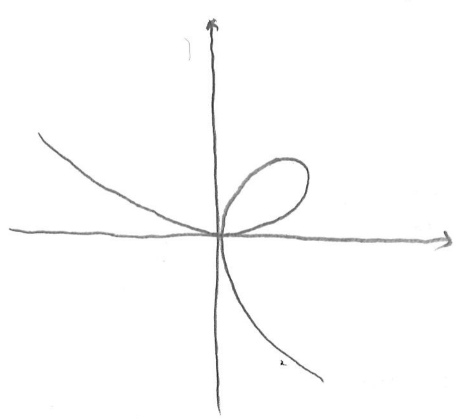
a. $x^2+y^2=25$
is a circle of radius 5
centered at $(0,0)$

$f(x) = \sqrt{25-x^2}$

$g(x) = -\sqrt{25-x^2}$

On the other hand it is really difficult to solve Equation 2 for y explicitly as a function (or several functions).

However equation 2 has a curve called the <u>Folium of Descartes</u>



Lec 12

However, we don't need to solve an equation for y in terms of x in order to find the derivative of y. Instead we use the method of <u>implicit differentiation</u>.

This involves differentiating both sides of the equation w.r.t x and solving the resulting equation for y'.

In this chapter the equation implies y is a differentiable function of x.

<u>Ex</u> If $x^2 + y^2 = 25$, Find $\frac{dy}{dx}$

<u>Soln</u> Differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Since y is a function of x, to find $\frac{d}{dx}(y^2)$ we use Chain rule,

$$\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}$$

So we have

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x = -2y\frac{dy}{dx} \Rightarrow \frac{2x}{-2y} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

<u>Ex</u> Find the tangent to the Folium of Descartes $x^3 + y^3 = 6xy$ at the pt $(3,3)$.

<u>Soln</u> Find $\frac{dy}{dx}$.

To do so differentiate both sides with respect to x

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$$

$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 6\frac{d}{dx}(xy)$$

$$\Rightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

$$\Rightarrow 3y^2 \cdot \frac{dy}{dx} - 6x\frac{dy}{dx} = -3x^2 + 6y$$

$$\Rightarrow 3\frac{dy}{dx}(y^2 - 2x) = 3(2y - x^2)$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Then slope of tangent line at $(3,3)$ is

$$m = \frac{2.3 - 3^2}{3^2 - 2.3} = \frac{6-9}{9-6} = -1$$

So the eqn of the tangent line is

$$y - 3 = -1(x-3) \Rightarrow y - 3 = -x + 3 \Rightarrow \boxed{x + y = 6}$$

Ex Find y' if $\sin(x+y) = y^2 \cos x$

Differentiate implicitly wrt x and remembering y is a function of x

$$\frac{d}{dx}\left[\sin(x+y)\right] = \frac{d}{dx}\left[y^2 \cos x\right]$$

$$\Rightarrow \cos(x+y) \cdot \frac{d}{dx}(x+y) = y^2 \cdot \frac{d}{dx}(\cos x) + \frac{d}{dx}(y^2) \cdot \cos x$$

$$\Rightarrow \cos(x+y) \cdot (1 + y') = y^2(-\sin x) + 2y \cdot y' \cdot \cos x$$

$$\Rightarrow \cos(x+y) + y' \cdot \cos(x+y) = -y^2 \sin x + y' \cdot (2y \cos x)$$

$\cos(x+y) + y^2 \sin x = y'(2y\cos x) - y'\cos(x+y)$

$\Rightarrow \cos(x+y) + y^2 \sin x = y'(2y\cos x - \cos(x+y))$

$$\Rightarrow \boxed{y' = \frac{\cos(x+y) + y^2\sin x}{2y\cos x - \cos(x+y)}}$$

Ex Find y'' if $x^4 + y^4 = 16$

Differentiating both sides wrt x, we have

$$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(16)$$

(*) $4x^3 + 4y^3 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x^3}{4y^3} = -\frac{x^3}{y^3}$

$$\frac{d}{dx}(4x^3 + 4y^3 y') = 0$$

$$12x^2 + \frac{d}{dx}(4y^3)\cdot y' + 4y^3 \frac{d}{dx}(y') = 0$$

$$\Rightarrow 12x^2 + 12y^2 \cdot y' \cdot y' + 4y^3 \cdot y'' = 0$$

$$\Rightarrow \frac{12x^2 + 12y^2 \cdot (y')^2}{4} = \frac{-4y^3 \cdot y''}{4}$$

$$\Rightarrow 3x^2 + 3y^2\left(\frac{-x^3}{y^3}\right)^2 = -y^3 \cdot y''$$

$$\Rightarrow 3x^2 + 3\cancel{y^2} \cdot \frac{x^6}{y^{\cancel{6}4}} = -y^3 \cdot y''$$

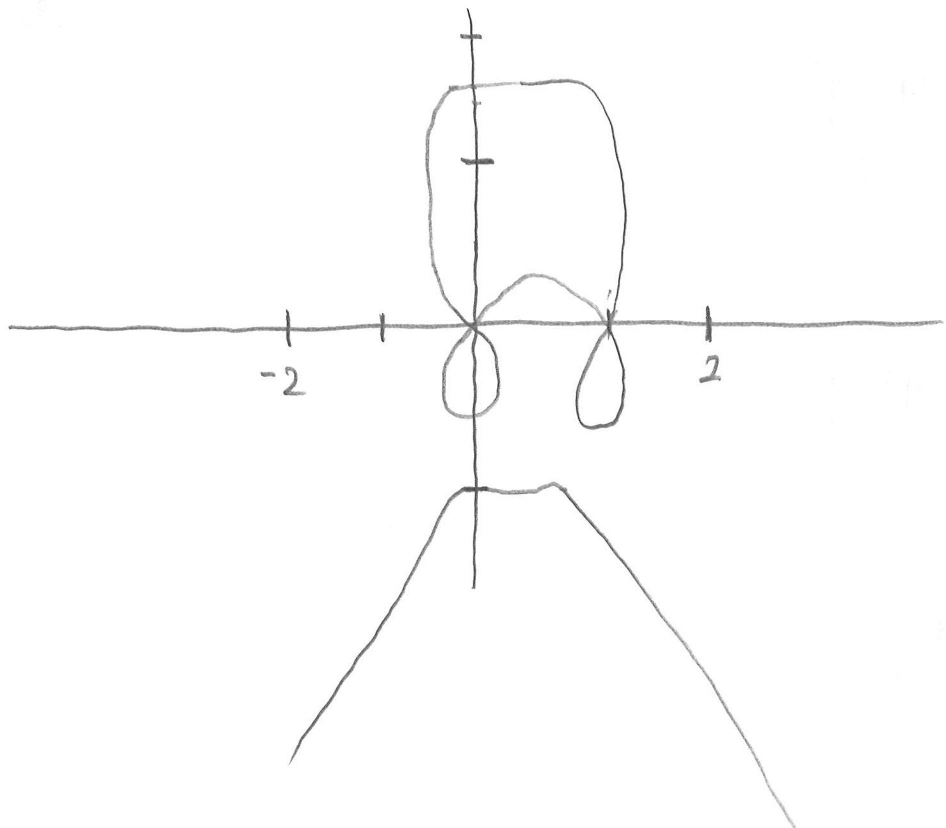
$$\Rightarrow 3x^2 + \frac{3x^6}{y^4} = -y^3 \cdot y''$$

$$\Rightarrow \frac{3x^2y^4 + 3x^6}{y^4} = -y^3 \cdot y''$$

$$\Rightarrow -\frac{3x^2y^4 + 3x^6}{y^7} = y''$$

$$\Rightarrow \frac{-3x^2(x^4 + y^4)}{y^7} = y''$$

$$\Rightarrow \boxed{\frac{-48x^2}{y^7} = y''}$$



$$2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$$ (Bouncing Wagon)

Find the values of x for which the graph have horizontal tangents ?

So first we want to find $\frac{dy}{dx} = y'$

$$\frac{d}{dx}\left(2y^3 + y^2 - y^5\right) = \frac{d}{dx}\left(x^4 - 2x^3 + x^2\right)$$

$$6y^2 \cdot y' + 2y \cdot y' - 5y^4 \cdot y' = 4x^3 - 6x^2 + 2x$$

$$\Rightarrow y'\left[6y^2 + 2y - 5y^4\right] = 4x^3 - 6x^2 + 2x$$

$$y' = \frac{4x^3 - 6x^2 + 2x}{6y^2 + 2y - 5y^4}$$

When $y' = 0$, we need to set numerator equal to 0 i.e.

$$4x^3 - 6x^2 + 2x = 0 \Rightarrow 2x(2x^2 - 3x + 1) = 0$$

$$\Rightarrow 2x = 0 \quad \text{or} \quad 2x^2 - 3x + 1 = 0$$

$$\frac{(2x-2)(2x-1)}{2} = 0$$

$$(x-1)(2x-1) = 0 \Rightarrow x = 1 \text{ or } 2x - 1 = 0$$

$$\Downarrow$$

$$x = \frac{1}{2}$$

Therefore horizontal tangent lines at

$$x = 0, \frac{1}{2}, 1$$